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ON UNIQUENESS AND CONTINUOUS DEPENDENCE IN DYNAMICAL PROBLEMS OF LINEAR THERMOELASTICITY

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1. INTRODUCTION

IN A series of recent papers [5–7] the present authors have studied questions of continuous dependence and uniqueness for solutions of various classes of initial boundary value problems in linear anisotropic elasticity. Logarithmic convexity arguments have been used in these investigations and in particular uniqueness has been established for the various classes of problems with no definiteness assumption on the energy.

In this paper the same convexity methods are used to examine the question of continuous dependence on the initial data of solutions to the linear anisotropic thermoelastic initial boundary value problem. Here, again, we are able to deduce uniqueness under rather weak assumptions on the coefficients in the governing equations.

Similar techniques have recently been used by Knops and Steel [8] to derive new uniqueness results for elastic mixtures.

2. STATEMENT OF PROBLEMS CONSIDERED

We assume that a linear anisotropic thermoelastic material occupies a closed bounded region B of three space with sufficiently smooth boundary ∂B. The governing equations have the following form (cf. [4]):

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)+\frac{\partial}{\partial x_j}(F_{ij}\theta) = \rho\mathscr{F}_i \tag{2.1}$$

and

$$\frac{\partial\theta}{\partial t}+cF_{ij}\frac{\partial^2 u_i}{\partial x_j\partial t} = \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial\theta}{\partial x_j}\right) \tag{2.2}$$

in $B\times(0, T]$,

where $u_i(\mathbf{x}, t)$ designates the cartesian components of displacement, $\theta(\mathbf{x}, t)$ is the temperature deviation and the $\mathscr{F}_i$ are the cartesian components of the prescribed body force per unit volume. In (2.1) and (2.2) the convention is adopted of summing over repeated suffixes whose range is 1, 2, 3. We are interested in the behaviour of the material during only the finite time interval $[0, T]$. Therefore we have restricted (2.1) and (2.2) to the cartesian product of the sets B and $(0, T]$. In the equations (2.1) and (2.2), the constant c is prescribed, while the elasticities $c_{ijkl}(\mathbf{x})$, the density $\rho(\mathbf{x})$, the conductivity tensor $a_{ij}(\mathbf{x})$ and the quantity $F_{ij}(\mathbf{x})$, assumed to be prescribed functions of $\mathbf{x}$ (x_1, x_2, x_3) alone, satisfy the following restrictions:

$$
\left.
\begin{array}{l}
\text{(a) } \rho \geq \rho_m > 0 \text{ for constant } \rho_m, \\
\text{(b) there exist finite constants } M_1, M_2 \text{ such that} \\
\qquad F_{ij}F_{ij} \leq M_1^2, \qquad \dfrac{\partial F_{ij}}{\partial x_j}\dfrac{\partial F_{ik}}{\partial x_k} \leq M_2^2, \\
\text{(c) } c_{ijkl} = c_{klij}, \qquad a_{ij} = a_{ji}, \\
\text{(d) } a_{ij} \text{ is positive-definite i.e. there exists a positive constant } a_0 \text{ such that} \\
\qquad a_{ij}\xi_i\xi_j \geq a_0\xi_i\xi_i
\end{array}
\right\} \tag{2.3}
$$

for all vectors ξ_i. (This last property accords with the Clausius–Duhem inequality.)

The type of initial boundary value problem considered here is that in which the temperature, displacement and velocity are initially prescribed. In addition, the temperature is prescribed on a portion $\bar{\sigma}$ of ∂B, the heat flux is given on $\partial B - \bar{\sigma}$, the displacement is prescribed on a portion $\bar{\Sigma}$ of ∂B, and the traction is given on $\partial B - \bar{\Sigma}$. We shall, however, limit our attention to the special cases:

$$
\begin{array}{ll}
& \text{(a)} \quad \partial B - \bar{\sigma} \quad \text{is empty} \\
\text{or} & \\
& \text{(b)} \quad \partial B - \bar{\Sigma} \quad \text{is empty}
\end{array} \tag{2.4}
$$

and further consider only classical solutions of (2.1) and (2.2), assuming in particular that the displacement and temperature are continuous in the closure of B. Naturally we could equally well treat classes of weak solutions to the above problem, as was done in [5].

Throughout this paper, we shall say that u_i and θ are solutions of problem $\mathscr{P}$ if they satisfy (2.1) and (2.2), the designated initial conditions and either boundary conditions of type (2.4a) or of type (2.4b). Further, the displacement is said to be of class $\mathcal{N}$ $(u_i \in \mathcal{N})$ if it satisfies the inequality

$$\int_0^T \int_{B(\eta)} \rho u_i u_i \,\mathrm{d}x\,\mathrm{d}\eta \leq N^2$$

for some prescribed (finite) constant N; here, and later, the symbol $B(t)$ means integration over the body B at time t.

If $c = 0$ then equations (2.1) and (2.2) become uncoupled in the sense that (2.2) is independent of u_i so that it reduces to the ordinary heat equation. The quantity $(\partial/\partial x_j)(F_{ij}\theta)$ appearing in (2.1) may then be regarded as an additional body force term so that the results of Knops and Payne [7] are applicable. Thus, without loss, we assume in the sequel that $c \neq 0$.

3. STABILITY UNDER PERTURBATIONS OF THE INITIAL DATA

Let (u_i^1, θ^1) denote a solution of problem $\mathscr{P}$, and (u_i^2, θ^2) a solution corresponding to the same body force $\mathscr{F}_i$ and boundary conditions but with different initial conditions. Without loss then we may consider solutions $u_i = (u_i^1 - u_i^2)$ and $\theta = (\theta^1 - \theta^2)$ of $\mathscr{P}$ with $\mathscr{F}_i \equiv 0$ and with homogeneous boundary data. This problem is designated by the symbol $\mathscr{P}_0$. The object of this paper is to examine the dependence on the initial data of solutions to $\mathscr{P}_0$. In fact, we establish the following theorem:

THEOREM 1. *If (u_i, θ) is a solution of $\mathscr{P}_0$ and $u_i \in \mathscr{N}$ then for finite time it is possible to determine explicit positive constants K_i such that*

$$\begin{aligned}\int_0^t \int_{B(\eta)} \rho u_i u_i \,\mathrm{d}x\,\mathrm{d}\eta \leq K_1 N_1^{2\delta} \Bigg\{ K_2 \int_{B(0)} \rho u_i u_i \,\mathrm{d}x + K_3 \int_{B(0)} \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}x \\ + K_4 \left| \int_{B(0)} c_{ijkl} \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\,\mathrm{d}x \right| + K_5 \int_{B(0)} \theta^2 \,\mathrm{d}x \Bigg\}^{1-\delta},\end{aligned} \tag{3.1}$$

where

$$\delta = \frac{1-\exp(-K_0 t)}{1-\exp(-K_0 T)}. \tag{3.2}$$

The proof of this theorem makes use of logarithmic convexity arguments similar to those employed already by the authors in treating the corresponding elastic problems [5–7]. Before proceeding with the proof, however, we first develop some auxiliary lemmas.

LEMMA 1. *If (u_i, θ) is a solution of $\mathscr{P}_0$ then*

$$\begin{aligned}\int_0^t \int_{B(\eta)} \theta^2 \,\mathrm{d}x\,\mathrm{d}\eta + 2\int_0^t \int_{B(\eta)} (t-\eta) a_{ij} \frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta \\ \leq 2t \int_{B(0)} \theta^2 \,\mathrm{d}x + 2[d_1 + d_2 t] \int_0^t \int_{B(\eta)} (t-\eta)\rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}x\,\mathrm{d}\eta,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\int_{B(t)} \theta^2 \,\mathrm{d}x + \int_0^t \int_{B(\eta)} a_{ij} \frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta \\ \leq (1+\nu t) \int_{B(0)} \theta^2 \,\mathrm{d}x + [(1+\nu t) d_1 + (\nu^{-1} + t^2) d_2] \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}x\,\mathrm{d}\eta,\end{aligned} \tag{3.4}$$

where

$$d_1 = \frac{M_1^2 c^2}{\rho_m a_0}, \qquad d_2 = \frac{2M_2^2 c^2}{\rho_m}, \tag{3.5}$$

and ν is an arbitrary positive constant.

To prove this lemma we observe that from (2.2) and an integration by parts it follows that

$$\begin{aligned}\int_0^t\int_{B(\eta)}\theta^2\,\mathrm{d}x\,\mathrm{d}\eta &= -\int_0^t\int_{B(\eta)}\frac{\partial}{\partial\eta}(t-\eta)\theta^2\,\mathrm{d}x\,\mathrm{d}\eta\\ &= t\int_{B(0)}\theta^2\,\mathrm{d}x+2\int_0^t\int_{B(\eta)}(t-\eta)\theta\frac{\partial\theta}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta\\ &= t\int_{B(0)}\theta^2\,\mathrm{d}x-2\int_0^t\int_{B(\eta)}(t-\eta)a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta\\ &\quad+2c\int_0^t\int_{B(\eta)}(t-\eta)\frac{\partial u_i}{\partial\eta}\frac{\partial}{\partial x_j}[F_{ij}\theta]\,\mathrm{d}x\,\mathrm{d}\eta.\end{aligned}\tag{3.6}$$

An application of the arithmetic–geometric mean inequality to the last term gives for arbitrary positive constants α_1 and α_2,

$$\begin{aligned}&2c\int_0^t\int_{B(\eta)}(t-\eta)\frac{\partial}{\partial x_j}[F_{ij}\theta]\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta\\ &\le |c|\rho_m^{-\frac{1}{2}}\left\{\alpha_1\int_0^t\int_{B(\eta)}\theta^2\,\mathrm{d}x\,\mathrm{d}\eta+\alpha_2\int_0^t\int_{B(\eta)}(t-\eta)a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta\right.\\ &\quad\left.+\left[\frac{M_2^2t}{\alpha_1}+\frac{M_1^2}{a_0\alpha_2}\right]\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta\right\}.\end{aligned}\tag{3.7}$$

Thus by choosing $|c|\rho_m^{-\frac{1}{2}}\alpha_2 = 1$ and $|c|\rho_m^{-\frac{1}{2}}\alpha_1 = \frac{1}{2}$ we find, upon insertion of (3.7) into (3.6) that inequality (3.3) results. An immediate consequence of (3.3) is the further inequality:

$$\begin{aligned}&\int_0^t\int_{B(\eta)}\theta^2\,\mathrm{d}x\,\mathrm{d}\eta+2\int_0^t\int_{B(\eta)}(t-\eta)a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta\\ &\qquad\le 2t\int_{B(0)}\theta^2\,\mathrm{d}x+2(d_1t+d_2t^2)\int_0^t\int_{B(\eta)}\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta,\end{aligned}\tag{3.8}$$

with d_1 and d_2 given by (3.5).

Just as in the derivation of (3.6) it follows easily that

$$\int_{B(t)}\theta^2\,\mathrm{d}x+2\int_0^t\int_{B(\eta)}a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta = \int_{B(0)}\theta^2\,\mathrm{d}x+2c\int_0^t\int_{B(\eta)}\frac{\partial u_i}{\partial\eta}\frac{\partial}{\partial x_j}[F_{ij}\theta]\,\mathrm{d}x\,\mathrm{d}\eta.\tag{3.9}$$

Again using the arithmetic–geometric mean inequality on the last term with appropriately chosen constants, and making use of (3.8), one obtains immediately the desired inequality (3.4) and the lemma is proved.

It is worth noting that if σ is not empty one could use, after an application of the arithmetic–geometric mean inequality, a bound of the type

$$\int_0^t\int_{B(\eta)}\theta^2\,\mathrm{d}x\,\mathrm{d}\eta \le \frac{1}{\lambda a_0}\int_0^t\int_{B(\eta)}a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta,\tag{3.9a}$$

where λ is the first eigenvalue in the corresponding fixed–free membrane eigenvalue problem for B. Using such an inequality, it would be possible for computable constant d_3

to derive an inequality of the form

$$\int_{B(t)} \theta^2 \, dx + \int_0^t \int_{B(\eta)} a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, dx \, d\eta \leq \int_{B(0)} \theta^2 \, dx + d_3 \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, dx \, d\eta, \tag{3.9b}$$

instead of (3.4). A combination of (3.9a) and (3.9b) would then give

$$\int_{B(t)} \theta^2 \, dx + \lambda a_0 \int_0^t \int_{B(\eta)} \theta^2 \, dx \, d\eta \leq \int_{B(0)} \theta^2 \, dx + d_3 \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, dx \, d\eta.$$

If σ is empty then, by (2.4), $\partial B - \bar{\Sigma}$ is empty, and since

$$\int_{B(t)} \theta \, dx = \int_{B(0)} \theta \, dx - c \int_0^t \int_{B(\eta)} \frac{\partial}{\partial x_j} (F_{ij}) \frac{\partial u_i}{\partial \eta} \, dx \, d\eta,$$

we can make use of Poincaré's inequality to obtain instead of (3.9b),

$$\begin{aligned} \int_{B(t)} \theta^2 \, dx + \int_0^t \int_{B(\eta)} a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, dx \, d\eta \leq \int_{B(0)} \theta^2 \, dx + d_4 t \left[\int_{B(0)} \theta \, dx \right]^2 \\ + (d_5 + d_6 t) \int_0^t \int_{B(\eta)} \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, dx \, d\eta, \end{aligned} \tag{3.9c}$$

for computable d_4, d_5 and d_6.

LEMMA 2. *If (u_i, θ) is a solution of $\mathscr{P}_0$ then the function*

$$J(t) = \int_{B(t)} \theta^2 \, dx + 2 \int_0^t \int_{B(\eta)} a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, dx \, d\eta + c \int_{B(t)} \left[\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right] dx \tag{3.10}$$

is a constant independent of time.

This result arises from the identity

$$\begin{aligned} 0 &= \int_0^t \int_{B(\eta)} \frac{\partial u_i}{\partial \eta} \left\{ \rho \frac{\partial^2 u_i}{\partial \eta^2} - \frac{\partial}{\partial x_j} \left(c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + \frac{\partial}{\partial x_j} [F_{ij} \theta] \right\} \, dx \, d\eta \\ &= E(t) - E(0) + \int_0^t \int_{B(\eta)} \frac{\partial u_i}{\partial \eta} \frac{\partial}{\partial x_j} [F_{ij} \theta] \, dx \, d\eta, \end{aligned} \tag{3.11}$$

where

$$E(t) = \tfrac{1}{2} \int_{B(t)} \left[\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right] dx. \tag{3.12}$$

Here, we have made use of the symmetry condition (2.3c), and an integration by parts. But if (3.11) and (3.9) are now combined we obtain

$$J(t) \equiv J(0), \tag{3.13}$$

the required equality.

Lemma 2 gives us a conservation law for the quantity $J(t)$. If $c > 0$ and

$$\int_{B(t)} c_{ijkl} \psi_{ij} \psi_{kl} \, dx \geq 0$$

for all tensors ψ_{ij}, then (3.10) yields immediately ordinary Liapounov stability in the

J-norm and we need proceed no further. (Indeed, this result closely resembles the linear version of one obtained by Ericksen [2, 3] in discussing thermoelastic stability.)

Otherwise, let us define

$$F(t) = \int_0^t \int_{B(\eta)} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta + (T-t) \int_{B(0)} \rho u_i u_i \, \mathrm{d}x + \gamma, \tag{3.14}$$

where the non-negative constant γ, dependent upon the data, is to be determined later, and let us proceed to establish the inequality (3.1) with the help of $F(t)$.

Now, differentiation of (3.14) yields

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \int_{B(t)} \rho u_i u_i \, \mathrm{d}x - \int_{B(0)} \rho u_i u_i \, \mathrm{d}x = 2 \int_0^t \int_{B(\eta)} \rho u_i \frac{\partial u_i}{\partial \eta} \, \mathrm{d}x \, \mathrm{d}\eta, \tag{3.15}$$

and

$$\frac{\mathrm{d}^2 F}{\mathrm{d}t^2} = 2 \int_{B(t)} \rho u_i \frac{\partial u_i}{\partial t} \, \mathrm{d}x = 4 \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, \mathrm{d}x \, \mathrm{d}\eta - 2Q, \tag{3.16}$$

where

$$\begin{aligned} Q &= 2 \int_0^t E(\eta) \, \mathrm{d}\eta + \int_0^t \int_{B(\eta)} u_i \frac{\partial}{\partial x_j} [F_{ij} \theta] \, \mathrm{d}x \, \mathrm{d}\eta - \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} \, \mathrm{d}x \\ &= 2tE(0) + (t/c) \int_{B(0)} \theta^2 \, \mathrm{d}x - \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} \, \mathrm{d}x - (1/c) \int_0^t \int_{B(\eta)} \theta^2 \, \mathrm{d}x \, \mathrm{d}\eta \\ &\quad - (2/c) \int_0^t \int_{B(\eta)} (t-\eta) a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, \mathrm{d}x \, \mathrm{d}\eta + \int_0^t \int_{B(\eta)} u_i \frac{\partial}{\partial x_j} [F_{ij} \theta] \, \mathrm{d}x \, \mathrm{d}\eta, \end{aligned} \tag{3.17}$$

according to (3.10). An application of the arithmetic–geometric mean and the Schwarz inequalities to the last term yields

$$\begin{aligned} Q &\le 2tE(0) + (t/c) \int_{B(0)} \theta^2 \, \mathrm{d}x + (3/2|c|) \int_0^t \int_{B(\eta)} \theta^2 \, \mathrm{d}x \, \mathrm{d}\eta \\ &\quad + (2/|c|) \int_0^t \int_{B(\eta)} (t-\eta) a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, \mathrm{d}x \, \mathrm{d}\eta - \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} \, \mathrm{d}x + (d_2/4|c|) \int_0^t \int_{B(\eta)} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta \\ &\quad + (d_1^{1/2}/|c|) \left\{ \int_0^t \int_{B(\eta)} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta \int_0^t \int_{B(\eta)} a_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, \mathrm{d}x \, \mathrm{d}\eta \right\}^{1/2}. \end{aligned} \tag{3.18}$$

Let us now make use of (3.3) and (3.4) to write

$$\begin{aligned} Q &\le 2tE(0) + (t/c) \int_{B(0)} \theta^2 \, \mathrm{d}x + (3/2|c|) \left\{ 2t \int_{B(0)} \theta^2 \, \mathrm{d}x + 2(d_1 + d_2 t) \int_0^t \int_{B(\eta)} (t-\eta) \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \right. \\ &\quad \left. \times \, \mathrm{d}x \, \mathrm{d}\eta \right\} - \int_{B(0)} \rho u_i \frac{\partial u_i}{\partial t} \, \mathrm{d}x + (d_2/4|c|) \int_0^t \int_{B(\eta)} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta + (d_1^{1/2}/|c|) \left\{ \int_0^t \int_{B(\eta)} \rho u_i u_i \times \mathrm{d}x \, \mathrm{d}\eta \right\}^{1/2} \\ &\quad \times \left[\left\{ (1+\nu t) \int_{B(0)} \theta^2 \, \mathrm{d}x \right\}^{1/2} + \{ (1+\nu t) \, d_1 \right. \\ &\quad \left. + (\nu^{-1} + t^2) \, d_2 \}^{1/2} \left\{ \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, \mathrm{d}x \, \mathrm{d}\eta \right\}^{1/2} \right], \end{aligned} \tag{3.19}$$

where in the last expression we have used the fact that for real numbers a_1 and a_2

$$(a_1^2+a_2^2)^{1/2} \leq |a_1|+|a_2|. \tag{3.20}$$

Let us rewrite (3.19) as

$$\begin{aligned} Q \leq Q_1 &+ \frac{3(d_1+d_2t)}{|c|}\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta+\left[\frac{d_2}{4|c|}+\frac{d_1}{2c^2}\right]\int_0^t\int_{B(\eta)}\rho u_iu_i\,\mathrm{d}x\,\mathrm{d}\eta \\ &+\frac{d_1^{1/2}}{|c|}\{(1+vt)d_1+(v^{-1}+t^2)d_2\}^{1/2}\left\{\int_0^t\int_{B(\eta)}\rho u_iu_i\,\mathrm{d}x\,\mathrm{d}\eta\int_0^t\int_{B(\eta)}\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta\right\}^{1/2}, \end{aligned} \tag{3.21}$$

with Q_1 being given by:

$$Q_1 = 2tE(0)+\left[t\{c^{-1}+(3/2|c|)\}+\left(\frac{1+vt}{2}\right)\right]\int_{B(0)}\theta^2\,\mathrm{d}x-\int_{B(0)}\rho u_i\frac{\partial u_i}{\partial t}\,\mathrm{d}x \tag{3.21a}$$

We shall further require in subsequent calculations the following two lemmas:

LEMMA 3. *If* (u_i,θ) *is a solution of* $\mathscr{P}_0$ *then*

$$\left|\frac{dF}{dt}\right| \leq \frac{dF}{dt}+2\int_{B(0)}\rho u_iu_i\,\mathrm{d}x. \tag{3.22}$$

This lemma follows trivially from the inequality

$$\left|\frac{dF}{dt}\right| \leq \int_{B(t)}\rho u_iu_i\,\mathrm{d}x+\int_{B(0)}\rho u_iu_i\,\mathrm{d}x.$$

LEMMA 4. *If* (u_i,θ) *is a solution of* $\mathscr{P}_0$ *then*

$$\begin{aligned} \int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta &\leq \frac{1}{2}\frac{dF}{\partial t}+\int_0^t \mathrm{e}^{A_2(t-\eta)}[Q_1-A_1\gamma]\,\mathrm{d}\eta+\tfrac{1}{2}A_2\,\mathrm{e}^{A_2t}[F(t)-\gamma] \\ &+\frac{A_1}{A_2}(\mathrm{e}^{A_2t}-1)F(t)+A_2^{-1}\left[\frac{A_2}{2}+\frac{A_1}{A_2}\right][(A_2t-1)\,\mathrm{e}^{A_2t}+1]\int_{B(0)}\rho u_iu_i\,\mathrm{d}x, \end{aligned} \tag{3.23}$$

where γ *is given by* (3.14), Q_1 *by* (3.21a) *and*

$$\left.\begin{aligned} A_1 &= \frac{1}{4c^2}\{d_1[(1+vT)d_1+(v^{-1}+T^2)d_2]+|c|d_2+2d_1\}, \\ A_2 &= \frac{3}{|c|}(d_1+d_2T). \end{aligned}\right\} \tag{3.24}$$

To prove this lemma we note from (3.16) and (3.21) that

$$\begin{aligned} \int_0^t\int_{B(\eta)}\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta &= \frac{1}{4}\frac{\mathrm{d}^2F}{\mathrm{d}t^2}+\frac{Q}{2} \\ &\leq \frac{1}{4}\frac{\mathrm{d}^2F}{\mathrm{d}t^2}+\frac{Q_1}{2}+\frac{3(d_1+d_2t)}{2|c|}\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta \\ &+\left(\frac{d_2}{8|c|}+\frac{d_1}{4c^2}\right)\int_0^t\int_{B(\eta)}\rho u_iu_i\,\mathrm{d}x\,\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{B(\eta)}\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta \\ &+\frac{1}{8}\frac{d_1}{c^2}[(1+vt)d_1+(v^{-1}+t^2)d_2]\int_0^t\int_{B(\eta)}\rho u_iu_i\,\mathrm{d}x\,\mathrm{d}\eta. \end{aligned} \tag{3.25}$$

Here we have used the arithmetic–geometric mean inequality on the last term of (3.21). Combining we have

$$\int_0^t\int_{B(\eta)} \rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta \le \frac{1}{2}\frac{\mathrm{d}^2F}{\mathrm{d}t^2}+Q_1+A_2(t)\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta$$
$$+A_1(t)\int_0^t\int_{B(\eta)}\rho u_iu_i\mathrm{d}x\,\mathrm{d}\eta, \tag{3.26}$$

where

$$\left.\begin{aligned} A_1(t) &= \frac{1}{4c^2}\{d_1[(1+vt)d_1+(v^{-1}+t^2)d_2]+|c|d_2+2d_1\} \le A_1,\\ A_2(t) &= \frac{3(d_1+d_2t)}{|c|} \le A_2. \end{aligned}\right\} \tag{3.27}$$

Upon replacing $A_1(t)$ and $A_2(t)$ by their respective upper bounds A_1 and A_2, we may easily solve (3.26) for the third term on the right to obtain

$$\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta \le \mathrm{e}^{A_2t}\left\{\frac{1}{2}\int_0^t \mathrm{e}^{-A_2\eta}\frac{d^2F}{d\eta^2}\,\mathrm{d}\eta+\int_0^t\mathrm{e}^{-A_2\eta}[Q_1-A_1\gamma]\,\mathrm{d}\eta\right.$$
$$\left.+A_1\int_0^t\mathrm{e}^{-A_2\eta}F(\eta)\,\mathrm{d}\eta\right\}. \tag{3.28}$$

On integrating the first and last terms by parts in opposite directions, we obtain in a straightforward way

$$\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta \le \frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}t}+\int_0^t\mathrm{e}^{A_2(t-\eta)}[Q_1-A_1\gamma]\,\mathrm{d}\eta-\frac{A_1}{A_2}F(t)$$
$$+\frac{A_1}{A_2}F(0)\,\mathrm{e}^{A_2t}+\left[\frac{A_2}{2}+\frac{A_1}{A_2}\right]\int_0^t\mathrm{e}^{A_2(t-\eta)}\frac{\mathrm{d}F}{\mathrm{d}\eta}\,\mathrm{d}\eta. \tag{3.29}$$

Again, since $\mathrm{d}F/\mathrm{d}t+\int_{B(0)}\rho u_iu_i\,\mathrm{d}x \ge 0$ it follows trivially that

$$\int_0^t\mathrm{e}^{A_2(t-\eta)}\frac{\mathrm{d}F}{\mathrm{d}\eta}\,\mathrm{d}\eta \le \mathrm{e}^{A_2t}[F(t)-F(0)]+\frac{1}{A_2}[(A_2t-1)\,\mathrm{e}^{A_2t}+1]\int_{B(0)}\rho u_iu_i\,\mathrm{d}x, \tag{3.30}$$

and insertion of (3.30) into (3.29) establishes the lemma. We note of course that $F(0)\ge\gamma$.

We are now ready to proceed with the proof of the theorem. Using (3.14), (3.15), (3.21) and (3.27) we form

$$F\frac{\mathrm{d}^2F}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2 \ge 4s^2-2FQ$$
$$\ge 4s^2-2F\left\{Q_1+A_2\int_0^t\int_{B(\eta)}(t-\eta)\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\,\mathrm{d}x\,\mathrm{d}\eta\right.$$
$$+\left(\frac{d_2}{4|c|}+d_1/2c^2\right)\int_0^t\int_{B(\eta)}\rho u_iu_i\,\mathrm{d}x\,\mathrm{d}\eta$$
$$\left.+\left[4A_1-\frac{d_2}{|c|}-2d_1/c^2\right]^{1/2}\left[\int_0^t\int_{B(\eta)}\rho u_iu_i\mathrm{d}x\,\mathrm{d}\eta\int_0^t\int_{B(\eta)}\rho\frac{\partial u_i}{\partial\eta}\frac{\partial u_i}{\partial\eta}\mathrm{d}x\,\mathrm{d}\eta\right]^{1/2}\right\}, \tag{3.31}$$

where

$$s^2 = \int_0^t \int_{B(\eta)} \rho u_i u_i \, \mathrm{d}x \, \mathrm{d}\eta \int_0^t \int_{B(\eta)} \rho \frac{\partial u_i}{\partial \eta} \frac{\partial u_i}{\partial \eta} \, \mathrm{d}x \, \mathrm{d}\eta - \left[\int_0^t \int_{B(\eta)} \rho u_i \frac{\partial u_i}{\partial \eta} \, \mathrm{d}x \, \mathrm{d}\eta \right]^2 \geq 0. \tag{3.32}$$

Now using lemma 4 it follows that

$$\begin{aligned} F\frac{\mathrm{d}^2 F}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2 &\geq 4s^2 - 2FI_2 - A_2 F \frac{\mathrm{d}F}{\mathrm{d}t} - \left(\frac{d_2}{2|c|} + \frac{d_1}{c_2}\right) F^2 - A_2 \left[A_2 + \frac{2A_1}{A_2}(1 - \mathrm{e}^{-A_2 T}) \right] \mathrm{e}^{A_2 T} F^2 \\ &\quad - \left[16A_1 - \frac{4d_2}{|c|} - \frac{8d_1}{c^2} \right]^{1/2} F \left\{ s^2 + \left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2 \right\}^{1/2}, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} I_2 &= Q_1 + \left[\frac{A_2}{2} + \frac{A_1}{A_2} \right] [(A_2 T - 1)\,\mathrm{e}^{A_2 T} + 1] \int_{B(0)} \rho u_i u_i \, \mathrm{d}x \\ &\quad + A_2 \int_0^t \mathrm{e}^{A_2(t-\eta)} [Q_1 - A_1 \gamma] \, d\eta - \left[\frac{d_2}{4|c|} + \frac{d_1}{2c^2} \right] \gamma - \tfrac{1}{2} A_2^2 \, \mathrm{e}^{A_2 T} \gamma. \end{aligned} \tag{3.34}$$

Clearly

$$\left[s^2 + \left(\frac{\mathrm{d}F}{\mathrm{d}t} \right)^2 \right]^{1/2} \leq s + \left| \frac{\mathrm{d}F}{\mathrm{d}t} \right| \leq s + \frac{\mathrm{d}F}{\mathrm{d}t} + 2 \int_{B(0)} \rho u_i u_i \, \mathrm{d}x, \tag{3.35}$$

the last inequality following from lemma 3.

Let us now take γ of the form

$$\gamma = \hat{k}_0 \int_{B(0)} \theta^2 \, \mathrm{d}x + \hat{k}_1 \int_{B(0)} \rho u_i u_i \, \mathrm{d}x + \hat{k}_2 \int_{B(0)} \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} \, \mathrm{d}x + \hat{k}_3 |E(0)|, \tag{3.36}$$

with the $\hat{k}_i$'s chosen so large that

$$I_2 + \left[16A_1 - \frac{4d_2}{|c|} - 8d_1/c^2 \right]^{1/2} \int_{B(0)} \rho u_i u_i \, \mathrm{d}x \leq 0. \tag{3.37}$$

This means that for fixed computable $\bar{k}$ we must have

$$\left\{ \tfrac{1}{2} A_2^2 \, \mathrm{e}^{A_2 T} + \frac{d_2}{4|c|} + \frac{d_1}{2c^2} \right\} \gamma \geq Q_1 + A_2 \int_0^t \mathrm{e}^{A_2(t-\eta)} Q_1(\eta) \, \mathrm{d}\eta + \bar{k} \int_{B(0)} \rho u_i u_i \, \mathrm{d}x, \tag{3.38}$$

which is clearly possible in view of the expression for Q_1.

Making use of (3.37) and (3.35) we now have

$$F \frac{\mathrm{d}^2 F}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}F}{\mathrm{d}t} \right)^2 \geq 4s^2 - \tilde{K}_1 F^2 - \tilde{K}_2 F s - \tilde{K}_3 F \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.39}$$

where

$$\left. \begin{aligned} \tilde{K}_1 &= A_2 \left[A_2 + \frac{2A_1}{A_2}(1 - \mathrm{e}^{-A_2 T}) \right] \mathrm{e}^{A_2 T} + \frac{d_2}{2|c|} + \frac{d_1}{c^2}, \\ \tilde{K}_2 &= \left[16A_1 - \frac{4d_2}{|c|} - \frac{8d_1}{c^2} \right]^{1/2}, \\ \tilde{K}_3 &= \tilde{K}_2 + A_2. \end{aligned} \right\} \tag{3.40}$$

The non-negative terms resulting from (3.37) have been dropped on the right. If we now complete the square on the right and again drop the non-negative term we obtain

$$F\frac{\mathrm{d}^2F}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)^2 \geq -2k_1F^2-k_2F\frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.41}$$

where

$$k_1 = \frac{16\tilde{K}_1+\tilde{K}_2^2}{32}, \qquad k_2 = \tilde{K}_3.$$

If $F(t)$ is identically zero for $0 \leq t \leq T$ the problem is of no interest. Hence, we assume that there exists an open interval (t_1, t_2) on which $F(t) > 0$. Then for $0 \leq t_1 < t < t_2 \leq T$ we may divide (3·41) by F^2 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}t}\right)+\frac{k_2}{F}\frac{\mathrm{d}F}{\mathrm{d}t}+2k_1 \geq 0, \tag{3.42}$$

or

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\ln F)+k_2\frac{\mathrm{d}}{\mathrm{d}t}(\ln F)+2k_1 \geq 0.$$

On setting

$$\tau = \mathrm{e}^{-k_2t},$$

we may write (3.42) as

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\{\ln(F\tau^{-2k_1/k_2^2})\} \geq 0, \tag{3.43}$$

where F is now to be regarded as a function of τ. Jensen's inequality together with the continuity of $F(\tau)$ then gives

$$F(\tau)\tau^{-2k_1/k_2^2} \leq [F(\tau_1)\tau_1^{-2k_1/k_2^2}]^{(\tau-\tau_2)/(\tau_1-\tau_2)}[F(\tau_2)\tau_2^{-2k_1/k_2^2}]^{(\tau_1-\tau)/(\tau_1-\tau_2)}, \tag{3.44}$$

where

$$\tau_1 = \mathrm{e}^{-k_2t_1}, \qquad \tau_2 = \mathrm{e}^{-k_2t_2}. \tag{3.45}$$

Now, either $F(t_1) = 0$ or $t_1 = 0$. If $F(t_1)$ vanishes then it follows from (3.44) that $F(t)$ vanishes identically for $t_1 \leq t \leq t_2$ and hence for $0 \leq t \leq T$. This fact clearly implies the uniqueness of the solution u_i to problem $\mathscr{P}_0$ under the assumptions of the theorem. Thus, without loss, we may take $t_1 = 0$ and so obtain from (3.44)

$$F(t)\,\mathrm{e}^{2k_1t/k_2} \leq [F(0)]^{(\mathrm{e}^{-k_2t}-\mathrm{e}^{k_2T})/(1-\mathrm{e}^{-k_2T})}[F(T)\,\mathrm{e}^{2k_1T/k_2}]^{(1-\mathrm{e}^{-k_2t})/(1-\mathrm{e}^{-k_2T})}. \tag{3.46}$$

Since $u_i \in \mathscr{N}$ it follows from (3.14) and (3.36) that if the initial temperature, displacement, and velocity are square integrable and the initial energy is finite then there exists a finite constant N_1 such that

$$F(T) \leq N_1^2. \tag{3.47}$$

Then immediately from (3.46) we have

$$\int_0^t \int_{B(\eta)} \rho u_i u_i \,\mathrm{d}x\,\mathrm{d}\eta + (T-t)\int_{B(0)} \rho u_i u_i \,\mathrm{d}x + \gamma \le \mathrm{e}^{2k_1[\delta T - t]/k_2} N_1^{2\delta}\left[T\int_{B(0)} \rho u_i u_i \,\mathrm{d}x + \gamma\right]^{1-\delta}, \tag{3.48}$$

with δ given by (3.2) (with K_0 replaced by k_2). Inequality (3.1) then follows with the help of (3.36), the arithmetic–geometric mean inequality and other simple devices.

4. EXTENSION OF RESULTS IN SECTION 3

The continuous dependence inequality of Theorem 1 does not directly imply the continuous dependence of θ in *any* appropriate norm on the initial data. However, in order to establish this conclusion we integrate (3.3) and (3.23) with respect to t to obtain respectively

$$\int_0^t \int_{B(\eta)} (t-\eta)\theta^2 \,\mathrm{d}x\,\mathrm{d}\eta + \int_0^t \int_{B(\eta)} (t-\eta)^2 a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta \le t^2\int_{B(0)} \theta^2 \,\mathrm{d}x$$
$$+ [d_1 + d_2 T]\int_0^t \int_{B(\eta)} (t-\eta)^2 \rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}x\,\mathrm{d}\eta, \tag{4.1}$$

and

$$\frac{1}{2}\int_0^t \int_{B(\eta)} (t-\eta)^2 \rho \frac{\partial u_i}{\partial \eta}\frac{\partial u_i}{\partial \eta}\,\mathrm{d}x\,\mathrm{d}\eta \le b_1 F + b_2\gamma, \tag{4.2}$$

where the b_i are computable positive constants. Substitution of (4.2) into (4.1) and use of (3.1) then clearly establishes the continuous dependence of θ in the norm

$$\|(\cdot)\|_t = \int_0^t \int_{B(\eta)} (t-\eta)\theta^2 \,\mathrm{d}x\,\mathrm{d}\eta + \frac{1}{2}\int_0^t \int_{B(\eta)} (t-\eta)^2 a_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}\eta \tag{4.3}$$

on the initial data measured by an appropriate norm.

Continuous dependence of θ in our L_2-norm may be obtained from (3.4) together with the inequality derived from (3.1) by replacing u_i with $\partial u_i/\partial t$. Now, however, the initial data must be such that the displacement, its first and second derivatives are all initially square integrable.

Finally, the results of this section together with Theorem 1 yield the following two corollaries:

COROLLARY 1. *There is at most one solution of the problem $\mathscr{P}$.*

A uniqueness theorem for the generalized solution of $\mathscr{P}$, based however on the stronger requirement of a definite energy, is proved by Dafermos [1].

COROLLARY 2. *In $B\times[0, T]$ the solution of the problem $\mathscr{P}$ depends continuously on the initial data (where both solution and data are measured in appropriate norms) provided the displacement u_i is of class $\mathscr{N}$.*

It is of mathematical interest to note that if (2.1) and (2.2) were of slightly different form, i.e.

$$\rho\frac{\partial^2 u_i}{\partial t^2}-\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)+F_{ij}\frac{\partial \theta}{\partial x_j}=\rho\mathscr{F}_i, \tag{2.1'}$$

and

$$\frac{\partial \theta}{\partial t}+c\frac{\partial}{\partial x_j}\left[F_{ij}\frac{\partial u_i}{\partial t}\right]=\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \theta}{\partial x_j}\right), \tag{2.2'}$$

the mathematical problem would have been somewhat simpler and in fact the requirement of uniform boundedness on the derivatives of F_{ij} could be removed altogether. Unfortunately the thermoelastic system is not of this form.

Note added in proof—Since completing this manuscript, the authors have become aware of an earlier proof of Corollary 1 (uniqueness) by L. BRUN, *C. r. hebd. Séanc. Acad. Sci., Paris* **261**, 2584–2587 (1965) and *Jnl Méc.* **8**, 167–192 (1969). Brun's approach, based upon reciprocity, is however entirely different to that adopted here.

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